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A-Acceptability of Derivatives of Rational Approximations to EXP(Z)

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The question of A-acceptability in regard to derivatives of $R_{m/n}$, the $[m/n]$ Padé approximation to the exponential, is examined for a range of values of m and n. It is proven that $R'_{n-1/n}$, $R'_{n/n}$, $R'_{n+1/n}$ and $R''_{n/n}$ are A-acceptable and that numerous other choices of m and n lead to non-A-acceptability. The results seem to indicate that the A-acceptability pattern of $R^{(k)}_{m/n}$ displays an intriguing generalization of the Wanner–Hairer–Nørsett theorem on the A-acceptability of $R_{m/n}$.

1. INTRODUCTION

The importance of Padé approximations to the exponential function has been acknowledged for a considerable time. A large number of papers have been published on different aspects of these functions. Properties like existence, convergence, loci of zeros and poles and lately, in connection with the numerical solution of stiff ordinary differential equations, A-acceptability, have been discussed in great detail.

Given a rational approximation R of order $p \geqslant 1$, satisfying

$$R(z) = e^z + cz^{p+1} + \mathcal{O}(z^{p+2}), \qquad c \neq 0,$$

it follows that the function $R^{(k)}(z) = d^k R(z)/dz^k$ is an approximation of order $p-k$ for every $0 \leqslant k \leqslant p-1$. A natural question arises regarding the properties of $R^{(k)}(z)$ as an approximation to the exponential.

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In the present paper we address ourselves to the A-acceptability of the derivatives of the Padé approximations $R_{m/n}(z)$:

$$R_{m/n}(z) = P_{m/n}(z)/Q_{m/n}(z), \qquad m, n \geqslant 0,$$

where

$$Q_{m/n}(z) = \lim_{\varepsilon \downarrow 0} {}_1F_1 \left[\begin{matrix} -n+\varepsilon; \\ -n-m+\varepsilon; \end{matrix} \quad -z \right] = \sum_{k=0}^{n} (-1)^k \frac{(n+m-k)!}{(n+m)!} \binom{n}{k} z^k$$
$$P_{m/n}(z) = Q_{n/m}(-z). \tag{1}$$

${}_1F_1$ is the confluent hypergeometric function,

$${}_1F_1 \left[\begin{matrix} \alpha; \\ \beta; \end{matrix} \quad z \right] = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(\alpha)_k}{(\beta)_k} z^k,$$

where the factorial symbol is defined by

$$(\alpha)_0 = 1,$$
$$(\alpha)_k = (\alpha)_{k-1}(\alpha+k-1) = \alpha(\alpha+1)\cdots(\alpha+k-1), \qquad k \geqslant 1.$$

A rational approximation R to $\exp(z)$ is said to be A-acceptable if $|R(z)| < 1$ for every $\operatorname{Re} z < 0$. It is known [5] that $R_{m/n}$ is A-acceptable if and only if $m \leqslant n \leqslant m+2$. It will be shown in the sequel that the differential approximations $R_{m/n}^{(k)}$ do not preserve this property.

It follows at once from the Cauchy integral formula that, subject to the A-acceptability of $R_{m/n}$,

$$|R_{m/n}^{(k)}(z)| \leqslant \frac{k!}{(-\operatorname{Re} z)^k}, \qquad z \in \mathbf{C}^- := \{z \in \mathbf{C} : \operatorname{Re} z < 0\}.$$

Hence $|R_{m/n}^{(k)}(z)| \leqslant 1$ if $\operatorname{Re} z \leqslant -(k!)^{1/k}$. This is hardly satisfactory. Furthermore, the order star theory of Wanner et. al. [5] is not very useful in the present context, since too many degrees of freedom exist in $R_{m/n}^{(k)}$. All this implies that the A-acceptability of $R_{m/n}^{(k)}$ ought to be studied by the classical approach, i.e., by using the maximal modulus theorem for analytic functions.

Our main result is that $R_{m/n}^{(k)}$ is A-acceptable for

$$m = n+1, \qquad k = 1;$$
$$m = n, \qquad k = 0, 1, 2;$$
$$m = n-1, \qquad k = 0, 1;$$
$$m = n-2, \qquad k = 0.$$

2. Preliminary Results

In order to show that $|R^{(k)}_{m/n}(it)| \leqslant 1$, $t \in \mathbf{R}$, for various choices of m, n and k, we must establish some properties of $Q_{m/n}(it)$.

LEMMA 1. *Given $m, n \geqslant 0$ it is true that*

$$|Q_{m/n}(it)|^2 = \sum_{k=0}^{n} a_k t^{2k}$$

where

$$a_{n-k} = \begin{cases} (-1)^k \dfrac{n!\,m!\,(m+k)!\,(n-m-1-k)!}{k!\,(n-k)!\,(n-m-1-2k)!\,((n+m)!)^2}: & \\ \qquad n \geqslant m+1,\ 0 \leqslant k \leqslant \left[\dfrac{n-m-1}{2}\right] & \\ 0: \qquad n \geqslant m+1,\ \left[\dfrac{n-m+1}{2}\right] \leqslant k \leqslant n-m & \\ \dfrac{n!\,m!}{((n+m)!)^2} \dbinom{m+2k-n}{k} \dfrac{(m+k)!}{(n-k)!}: & \\ \qquad \max\{0, n-m\} \leqslant k \leqslant n. & \end{cases} \tag{2}$$

Proof. The substitution of (1) with $z = it$ into the Ramanujan formula [4, p. 106] gives

$$|Q_{m/n}(it)|^2 = \lim_{\varepsilon \downarrow 0} {}_2F_3 \left[\begin{matrix} n, -m+\varepsilon; \\ -n-m+\varepsilon, -\frac{1}{2}(n+m-\varepsilon), \frac{1}{2}(1-n-m+\varepsilon); \end{matrix} \; -\tfrac{1}{4}t^2 \right].$$

The expression (2) now follows by a straightforward manipulation of the factorial symbols. ∎

We set

$$H_n(t) := |Q_{n/n}(it)|^2, \qquad t \in \mathbf{R};$$

$$G_n(t) := -\frac{t}{2n-1} \operatorname{Re}\{Q_{n/n}(-it)\, Q_{n-1/n-1}(it)\}, \qquad t \in \mathbf{R}.$$

It follows at once from Lemma 1 that

$$H_n(t) = \left(\frac{n!}{(2n)!}\right)^2 \sum_{k=0}^{n} \frac{(2n-k)!}{k!} \binom{2n-2k}{n-k} t^{2k}.$$

LEMMA 2. *For every $n \geq 0$ it holds that*

$$G_n(t) = -2\mathrm{Im}\{Q_{n/n}(-it)\,Q_{n/n-1}(it)\} = -2H_n'(t)$$

and

$$\mathrm{Im}\{Q_{n/n}(-it)\,Q_{n-1/n-1}(it)\} = 2(2n-1)\left(\frac{n!}{(2n)!}\right)^2 t^{2n-1}.$$

Proof. Substitution of the identities

$$Q'_{n/n}(z) = -\tfrac{1}{2}Q_{n/n-1}(z)$$

and

$$Q_{n/n-1}(z) = Q_{n/n}(z) + \frac{z}{2(2n-1)}\,Q_{n-1/n-1}(z)$$

into

$$H_n'(t) = -2\mathrm{Im}\{Q'_{n/n}(it)\,Q_{n/n}(-it)\}$$

gives

$$H_n'(t) = \frac{t}{2(2n-1)}\,\mathrm{Re}\{Q_{n/n}(-it)\,Q_{n-1/n-1}(it)\} = -2H'_n(t).$$

The second relation is obtained by induction from

$$\begin{aligned}
&\mathrm{Im}\{Q_{n/n}(-it)\,Q_{n-1/n-1}(it)\}\\
&= \mathrm{Im}\left\{\left(Q_{n-1/n-1}(-it) - \frac{1}{4(2n-1)(2n-3)}\,t^2 Q_{n-2/n-2}(-it)\right) Q_{n-1/n-1}(it)\right\}\\
&= -\frac{t^2}{4(2n-1)(2n-3)}\,\mathrm{Im}\{Q_{n-1/n-1}(it)\,Q_{n-2/n-2}(-it)\}. \quad ∎
\end{aligned}$$

We note that the expression for $G_n(t)$ was already given in Theorem A.1 of Ehle and Picel [2].

3. THE $\mathscr{A}$-ACCEPTABILITY RESULTS

Straightforward differentiation yields

$$R'_{m/n}(z) = R_{m/n}(z) - \frac{P_{m/n}(z)\,Q_{m/n}(z) + P_{m/n}(z)\,Q'_{m/n}(z) - P'_{m/n}(z)\,Q_{m/n}(z)}{Q^2_{m/n}(z)}. \quad (3)$$

We now use the identities

$$P'_{m/n}(z)=\frac{m}{m+n}\,P_{m-1/n}(z),$$

$$Q'_{m/n}(z)=-\frac{n}{m+n}\,Q_{m/n-1}(z),$$

$$Q_{m/n}(z)=\frac{m}{m+n}\,Q_{m-1/n}(z)+\frac{n}{m+n}\,Q_{m/n-1}(z)$$

to show that the numerator in (3) equals

$$\frac{m}{m+n}\,(P_{m/n}(z)\,Q_{m-1/n}(z)-P_{m-1/n}(z)\,Q_{m/n}(z)).$$

The polynomial $P_{m/n}Q_{m-1/n}-P_{m-1/n}Q_{m/n}$ is of degree $n+m$ and, since the $[m/n]$th Padé approximation satisfies $R_{m/n}(z)-\exp(z)=\mathcal{O}(z^{m+n+1})$, it follows that

$$P_{m/n}(z)\,Q_{m-1/n}(z)-P_{m-1/n}(z)\,Q_{m/n}(z)=\mathcal{O}(z^{n+m}).$$

Therefore the polynomial equals dz^{n+m}. The constant d can be easily found from (1), giving

$$R'_{m/n}(z)=R_{m/n}(z)-(-1)^n\,\frac{m!\,n!}{((m+n)!)^2}\,\frac{z^{n+m}}{Q^2_{m/n}(z)}. \tag{4}$$

Let

$$S_{m/n}(z;\gamma):=R_{m/n}(z)-\gamma c_{m/n}\,\frac{z^{n+m}}{Q^2_{m/n}(z)},$$

where $\gamma\in\mathbf{R}$ and

$$c_{m/n}:=(-1)^n\,\frac{m!\,n!}{((m+n)!)^2}.$$

Because of (4) it holds that $S_{m/n}(z;1)\equiv R'_{m/n}(z)$.

Our intention being to use the maximal modulus theorem in $\mathbf{C}^-$, we need to examine whether $|S_{m/n}(it;\gamma)|^2\leqslant 1$ for every $t\in\mathbf{R}$. This is equivalent to

$$B_{m/n}(t;\gamma):=H_{m/n}(t)\,E_{m/n}(t)+2\gamma c_{m/n}t^{n+m}\,\mathrm{Re}\{(-i)^{n+m}\,P_{m/n}(it)\,Q_{m/n}(it)\}$$
$$-\gamma^2c^2_{m/n}t^{2(n+m)}\geqslant 0,\qquad t\in\mathbf{R}, \tag{5}$$

where $E_{m/n}(t)$ is the E-polynomial [3] of the $[m/n]$ Padé approximation

$$\begin{aligned} E_{m/n}(t) &:= |Q_{m/n}(it)|^2 - |P_{m/n}(it)|^2 \\ &= |Q_{m/n}(it)|^2 - |Q_{n/m}(it)|^2 \end{aligned}$$

and

$$H_{m/n}(t) := |Q_{m/n}(it)|^2.$$

THEOREM 3. *The approximation* $S_{n/n}(\cdot;\gamma)$ *is* A*-acceptable if and only if* $0 \leqslant \gamma \leqslant 2$.

Proof. Since all the poles of $R_{n/n}$ are in $\mathbf{C}^+ := \{z \in \mathbf{C}: \operatorname{Re} z > 0\}$ [5], $S_{n/n}(\cdot;\gamma)$ is analytic in $\mathbf{C}^-$ for every γ. Hence it just remains to verify that (5) holds if and only if $0 \leqslant \gamma \leqslant 2$.

By virtue of $E_{n/n}(t) \equiv 0$ we have

$$B_{n/n}(t;\gamma) = 2\gamma\, |c_{n/n}|\, t^{2n}\{H_{n/n}(t) - \tfrac{1}{2}\gamma\, |c_{n/n}|\, t^{2n}\}.$$

Therefore A-acceptability implies $\gamma \geqslant 0$, since $H_{n/n}(t) > 0$ for every $t \in \mathbf{R}$. Moreover, it follows from (2) that

$$\begin{aligned} H_{n/n}(t) - \frac{1}{2}\gamma\, |c_{n/n}|\, t^{2n} = \left(\frac{n!}{(2n)!}\right)^2 &\left\{\sum_{k=0}^{n-1} \binom{2(n-k)}{n-k} \frac{(2n-k)!}{k!}\, t^{2k}\right. \\ &\left. + \left(1 - \frac{1}{2}\gamma\right) t^{2n}\right\}. \end{aligned}$$

This completes the proof of the lemma, by ascertaining that A-acceptability occurs just for $0 \leqslant \gamma \leqslant 2$. ∎

COROLLARY. $R'_{n/n}$ *is* A*-acceptable.*

Proof. By setting $\gamma = 1$ in Theorem 3. ∎

We now turn our attention to $m = n-1$. A straightforward computation in (2) gives

$$H_{n-1/n}(t) = \frac{n!\,(n-1)!}{((2n-1)!)^2} \left\{\sum_{k=0}^{n-1} \binom{2(n-k)-1}{n-k} \frac{(2n-k-1)!}{k!}\, t^{2k} + \frac{1}{n}\, t^{2n}\right\} \tag{6}$$

and

$$E_{n-1/n}(t) = H_{n-1/n}(t) - H_{n/n-1}(t) = \left(\frac{(n-1)!}{(2n-1)!}\right)^2 t^{2n}. \tag{7}$$

The term $\operatorname{Re}\{(-i)^{2n-1} P_{n-1/n}(it)\, Q_{n/n-1}(it)\}$ is simplified by using the identities

$$P_{n-1/n}(it) = P_{n/n}(it) - \frac{it}{2(2n-1)} P_{n-1/n-1}(it),$$

$$Q_{n-1/n}(it) = Q_{n/n}(it) - \frac{it}{2(2n-1)} Q_{n-1/n-1}(it).$$

It follows that

$$\begin{aligned}
&\operatorname{Re}\{(-i)^{2n-1} P_{n-1/n}(it)\, Q_{n-1/n}(it)\} \\
&\quad = (-1)^{n-1} \operatorname{Im}\{P_{n-1/n}(it)\, Q_{n-1/n}(it)\} \\
&\quad = (-1)^{n-1} \operatorname{Im}\left\{\left(P_{n/n}(it) - \frac{it}{2(2n-1)} P_{n-1/n-1}(it)\right)\right. \\
&\qquad \left. \times\left(Q_{n/n}(it) - \frac{it}{2(2n-1)} Q_{n-1/n-1}(it)\right)\right\} \\
&\quad = \frac{(-1)^n}{2n-1}\, t \operatorname{Re}\{Q_{n/n}(-it)\, Q_{n-1/n-1}(it)\} = 2(-1)^n H_n'(t), \qquad (8)
\end{aligned}$$

the last identity being obtained by envoking Lemma 2. Equations (5)–(8) now give

$$\begin{aligned}
B_{n-1/n}(t) = n\left(\frac{(n-1)!}{(2n-1)!}\right)^4 t^{2n} &\left\{\sum_{k=0}^{n-2} \binom{2n-2k-2}{n-k-1}\right. \\
&\times \frac{(2n-k-1)!}{k!}\left(\gamma + 1 - \frac{1}{2(n-k)}\right) t^{2k} \\
&\left. + n(1+2\gamma-\gamma^2)\, t^{2(n-1)} + \frac{1}{n}\, t^{2n}\right\}.
\end{aligned}$$

The coefficients of t^{2k}, $0 \leqslant k \leqslant n-2$, are all non-negative if and only if $\gamma \geqslant -\frac{3}{4}$, whereas the coefficient of $t^{2(n-1)}$ is non-negative if $1-\sqrt{2} \leqslant \gamma \leqslant 1+\sqrt{2}$. Since all the zeros of $Q_{n-1/n}$ lie in the right half plane [1], the following theorem is true:

THEOREM 4. *The approximation $S_{n-1/n}(\cdot;\gamma)$ is A-acceptable if* $1-\sqrt{2} \leqslant \gamma \leqslant 1+\sqrt{2}$.

It now follows at once by setting $\gamma = 1$ in the last theorem that

COROLLARY. *$R'_{n-1/n}$ is A-acceptable.*

So far, the A-acceptability of the investigated Padé approximations and their derivatives was similar. The situation drastically changes regarding other choices of m and n.

THEOREM 5. *The approximation $S_{n/n-1}(\cdot;\gamma)$ is A-acceptable if and only if $\gamma=1$.*

Proof. It follows from (2) that

$$E_{n/n-1}(t) = -\left(\frac{(n-1)!}{(2n-1)!}\right)^2 t^{2n}$$

and

$$H_{n/n-1}(t)=\frac{n!(n-1)!}{((2n-1)!)^2}\sum_{k=0}^{n-1}\binom{2(n-k)-1}{n-k-1}\frac{(2n-k-1)!}{k!}t^{2k}.$$

A calculation similar to (8) gives

$$\operatorname{Re}\{(-i)^{2n-1}P_{n/n-1}(it)\,Q_{n/n-1}(it)\}=2(-1)^{n-1}H'_n(t).$$

Substitution in (5) yields

$$B_{n/n-1}(t)=n\left(\frac{(n-1)!}{(2n-1)!}\right)^4 t^{2n}\left\{\sum_{k=0}^{n-2}\binom{2(n-k-1)}{n-k-1}\frac{(2n-k-1)!}{k!}\right.$$
$$\left.\times\left(2\gamma-2+\frac{1}{n-k}\right)t^{2k}-n(1-\gamma)^2\,t^{2(n-1)}\right\}.$$

Hence $B_{n/n-1}(t)\geqslant 0$ for $t\geqslant 0$ if and only if $\gamma=1$. It is easy to see that this choice of γ gives non-negative coefficients in $B_{n/n-1}$ and so $|S_{n/n-1}(it;1)|\leqslant 1$ for every $t\in\mathbf{R}$.

The proof of A-acceptability for $\gamma=1$ is completed by noting that the zeros of $Q_{n/n-1}$ lie in $\mathbf{C}^+$; this follows from [5], since the zeros of $Q_{n/n-1}$ are mirror images, with respect to $i\mathbf{R}$, of the zeros of $P_{n-1/n}$. ∎

COROLLARY. *$R'_{n/n-1}$ is A-acceptable.*

Next we consider the A-acceptability of $R''_{n/n}$. It follows from (4) that

$$R''_{n/n}(z)=R_{n/n}(z)-(-1)^n c_n^2\,\frac{z^{2n-1}}{Q^3_{n/n}(z)}\,\Phi_n(z),$$

where $c_n = c_{n/n}$ and

$$\Phi_n(z) = (z+2n)\, Q_{n/n}(z) + zQ_{n/n-1}(z)$$
$$= 2(z+n)\, Q_{n/n}(z) + \frac{z^2}{2(2n-1)}\, Q_{n-1/n-1}(z).$$

THEOREM 5. *$R''_{n/n}$ is A-acceptable.*

Proof. Since the analyticity of $R''_{n/n}$ in $\mathbf{C}^-$ is a consequence of the A-acceptability of $R_{n/n}$, it sufficient to verify that $|R''_{n/n}(it)| \leqslant 1$ for every real t. This is equivalent to $F_n(t) \geqslant 0$, $t \in \mathbf{R}$, where

$$F_n(t) := 2H_n(t)\, \mathrm{Im}\{P_{n/n}(it)\, \Phi_n(it)\}/t - c_n^2 t^{2n-2}\, |\Phi_n(it)|^2.$$

Lemma 2 gives

$$\mathrm{Im}\{P_{n/n}(it)\, \Phi_n(it)\} = 2tH_n(t) - c_n^2 t^{2n+1}.$$

Furthermore, that lemma gives a useful expression for $|\Phi_n(it)|^2$,

$$|\Phi(it)|^2 = 4(t^2+n^2)\, H_n(t) - \frac{2t^2}{2n-1}\, \{t\, \mathrm{Im}\{Q_{n/n}(-it)\, Q_{n-1/n-1}(it)\}$$
$$+ n\, \mathrm{Re}\{Q_{n/n}(-it)\, Q_{n-1/n-1}(it)\}\} + \frac{t^4}{4(2n-1)^2}\, H_{n-1}(t)$$
$$= 4(t^2+n^2)\, H_n(t) - 4ntH'_n(t) + \frac{t^4}{4(2n-1)^2}\, H_{n-1}(t) - 4c_n^2 t^{2n+2}.$$

Thus, F_n has the form

$$F_n(t) = 4H_n^2(t) - (6t^2+4n^2)\, c_n^2 t^{2n-2} H_n(t) + 4nc_n^2 t^{2n-1} H'_n(t)$$
$$- \frac{c_n^2}{4(2n-1)^2}\, t^{2n+2} H_{n-1}(t) + 4c_n^4 t^{4n}.$$

Note that the coefficient of t^{4n} is $c_n^4 > 0$.

F_n is a quadratic in H_n, with the discriminant

$$D_n(t) = (3t^2+2n^2)^2\, c_n^4 t^{4n-4} - 4nc_n^2 t^{2n-1} H'_n(t) + \frac{c_n^2}{4(2n-1)}\, t^{2n+2} H_{n-1}(t)$$
$$- 4c_n^4 t^{4n} := t^{2n} \sum_{j=0}^{n} d_j t^{2j}.$$

It is easily ascertained that, given $n \geqslant 2$, $d_j < 0$ for every $0 \leqslant j \leqslant n$. Therefore $D_n(t) < 0$, $t \in \mathbf{R}$, and F_n, as a function of H_n, does not change sign. Since $F_n(t) > 0$ for $|t| \gg 0$, it follows that F_n is non-negative for every $t \in \mathbf{R}_n$ and

$n \geqslant 2$. A straightforward calculation verifies that $F_1(t) \geqslant 0$, $t \in \mathbf{R}$. This completes the proof of the theorem. ∎

We now proceed to investigate the A-acceptability problem from the other end, showing that a whole range of derivatives of Padé approximations cannot be A-acceptable.

Given

$$R(z) = e^z + Cz^{p+1} + \mathcal{O}(z^{p+2})$$

it follows by induction that for every $0 \leqslant k \leqslant p$

$$R^{(k)}(z) = e^z + C_k z^{p+1-k} + \mathcal{O}(z^{p+2-k}), \tag{9}$$

where

$$C_k = \frac{(p+1)!}{(p+1-k)!} C. \tag{10}$$

The important observation is that C_k has the same sign as C.

THEOREM 6. *If $n+m-k$ is odd and the numbers n and $[(n+m-k-1)/2]$ have the same parity then $R_{m/n}^{(k)}(z)$ cannot be A-acceptable.*

Proof. We use the order star theory [5], considering the order star of $R_{m/n}^{(k)}(z)$. The order is $p = n+m-k$. Since $p+1$ equi-angular fingers of the order star approach the origin, separated by equi-angular dual fingers, the imaginary axis in the neighbourhood of the origin (a) separates between the order star and the dual order star if p is even, (b) bisects a dual finger if either $C_k > 0$ and $[(p-1)/2]$ is even or $C_k < 0$ and $[(p-1)/2]$ is odd, and (c) bisects a finger if either $C_k > 0$ and $[(p-1)/2]$ is odd or $C_k < 0$ and $[(p-1)/2]$ is even. The last case leads to non-A-acceptability, since the order star intersects $i\mathbf{R}$.

The error constant of $R_{m/n}$ being

$$C = (-1)^{n-1} \frac{n!\,m!}{(n+m)!\,(n+m+1)!},$$

(9) and (10) imply that $(-1)^{n-1} C_k > 0$, $0 \leqslant k \leqslant n+m$.

The theorem now follows by introspection. ∎

COROLLARY. *Given $0 \leqslant n \leqslant m+3$,*

$$R_{m/n}^{(m-n+3)}(z)$$

is not A-acceptable.

In particular, we find that $R'_{n-2/n}$, $R''_{n-1/n}$, and $R'''_{n/n}$ are not A-acceptable.

Let R_k denote the kth derivative of R,

$$R_k(z) = \frac{P_k(z)}{Q^{k+1}(z)},$$

where $R = P/Q$ and $\deg P_k = m_k$. It follows easily by induction that, given $\deg P = m$, $\deg Q = n \leqslant m$, $1 \leqslant n \leqslant m-1$, $m_k = m + k(n-1)$ for $0 \leqslant k \leqslant m-n-1$, whereas $m_k = m + k(n-1) - 1$ for $m - n \leqslant k$.

LEMMA 7. *If* $1 \leqslant n \leqslant m-1$ *and* $0 \leqslant k \leqslant m-n-1$ *then* $R^{(k)}_{m/n}(z)$ *is not A-acceptable.*

Proof. Follows at once, since for every $0 \leqslant k \leqslant m-n-1$, $m_k > (k+1)n$ and the approximation is unbounded in $\mathbf{C}^-$. ∎

Based on these results we put forward the following conjecture:

Conjecture. The approximation $R^{(k)}_{m/n}(z)$ is A-acceptable if and only if either $n = 0$, $k \geqslant m$ or $1 \leqslant n \leqslant m+2$ and $\max\{0, m-n\} \leqslant k \leqslant m-n+2$.

In other words, we conjecture that the A-acceptable derivatives of Padé approximations can be ordered for every $n \geqslant 2$ in an array of the form

$$\begin{array}{lllll}
R_{n-2/n} & & & & \\
R_{n-1/n} & R'_{n-1/n} & & & \\
R_{n/n} & R'_{n/n} & R''_{n/n} & & \\
 & R'_{n+1/n} & R''_{n+1/n} & R'''_{n+1/n} & \\
 & & R''_{n+2/n} & R'''_{n+2/n} & R^{(\mathrm{IV})}_{n+2/n} \\
 & & \ddots & \ddots & \ddots
\end{array}$$

It is known [5] that exactly three A-acceptable Padé approximations exist for every $n \geqslant 2$. Our conjecture, if true, shows that the derivatives show a similar behaviour, albeit with different triplets.

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